Exact Solutions of the N-dimensional Radial Schrödinger Equation with the Coulomb Potential via the Laplace Tranform

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The second-order N-dimensional Schrödinger differential equation with the Coulomb potential is reduced to a first-order differential equation by means of the Laplace transform and the exact bound state solutions are obtained. It is shown that this method solving the Schrödinger equation may serve as a substitute for the factorization approach also in lower dimensions. — PACS numbers: 03.65.Ge.

Key words: Bound State; Coulomb Potential; Laplace Transforms.

Exact solutions of the Schrödinger equation with various potentials play an important role for molecules, atoms, nuclei, etc. In modern theory, also higher spatial dimensions are of interest. Recently, the high dimensional Schrödinger equation with the Coulomb potential has been discussed by many authors [1 – 7]. In this article, we employ the Laplace transform to solve the N-dimensional Schrödinger equation with the Coulomb potential. By Laplace transform the Schrödinger equation, which is a second-order differential equation, can be reduced to a first-order differential equation. From this the exact bound state solutions are obtained. It is shown that this method to solve the Schrödinger equation is easier than the previous known ones and may serve as a substitute for the factorization approach [8] also in lower dimensions. The Laplace transform, which is an integral transform, is very useful in physics and engineering [9]. Such technique was already used by Schrödinger in his first paper on the quantum mechanical hydrogen atom [10] and later by Swainson *et al.* on the recurrence relations of the radial wave functions for the hydrogen atom [11]. Recently, the exact bound state solutions of the Schrödinger equation with the Morse potential were also obtained [12].

With the motion of a particle in an N-dimensional Euclidian space, the time independent radial Schrödinger equation for any integral dimension is shown below [1 – 7]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{L(L+N-2)}{r^2} -\frac{2m}{\hbar^2}V(r)+\frac{2m}{\hbar^2}E\right]R_L(r)=0. \tag{1}$$

For the Coulomb potential $V(r) = -Ze^2/r$ Eq. (1) turns into

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{L(L+N-2)}{r^2} +\frac{2mZe^2}{\hbar^2 r}+\frac{2m}{\hbar^2}E\right]R_L(r)=0. \tag{2}$$

If one introduces

$$\alpha=\frac{2mZe^2}{\hbar^2}, \tag{3}$$

$$\beta^2=-\frac{2mE}{\hbar^2}, \tag{4}$$

$$R_L(r)=r^{-(L+N-2)}f_L(r), \tag{5}$$

Eq. (2) becomes

$$\left[r\frac{\mathrm{d}^2}{\mathrm{d}r^2}-(2L+N-3)\frac{\mathrm{d}}{\mathrm{d}r}-\beta^2 r+\alpha\right]f_L(r)=0. \tag{6}$$

The Laplace transforms [9] leads to the equation

$$(p^2-\beta^2)\frac{\mathrm{d}}{\mathrm{d}p}F_L(p)+[(2L+N-1)p-\alpha]F_L(p)=0. \tag{7}$$

Eq. (7) is a first-order differential equation and therefore one may directly make use of its integral to get the expression

$$F_L(p)=C''(p+\beta)^{-(2L+N-1)}\left(1-\frac{2\beta}{p+\beta}\right)^{\frac{\alpha-(2L+N-1)\beta}{2\beta}}, \tag{8}$$

where C'' is a constant. Noting that for non-integer exponent $\left(1-\frac{2\beta}{p+\beta}\right)^{\frac{\alpha-(2L+N-1)\beta}{2\beta}}$ is a multi-valued function and the wave-functions are required to be single-valued, we must take

$$\frac{\alpha-(2L+N-1)\beta}{2\beta}=n, \quad n=0,1,2,3,\ldots. \tag{9}$$

Applying a simple series expansion in powers of $(p+\beta)$ to Eq. (8) yields

$$F(p)=C'\sum_{j=0}^{n}\frac{(-2\beta)^j n!}{(n-j)!j!}(p+\beta)^{-(2L+N-1+j)}, \tag{10}$$

where C' is a constant. In terms of a simple extension of the inverse Laplace transforms [9] we can immediately deduce that

$$f_L(r)=Cr^{2L+N-2}e^{-\beta r} \cdot \sum_{j=0}^{n}\frac{(-1)^j n!\Gamma(2L+N-1)}{(n-j)!j!\Gamma(2L+N-1+j)}(2\beta r)^j. \tag{11}$$

where C is a constant. Comparing Eq. (11) with the series expansion of the confluent hypergeometric functions

$$F(-n,\gamma,z)=\sum_{j=0}^{n}\frac{(-1)^j n!\Gamma(\gamma)}{(n-j)!\Gamma(\gamma+j)}z^j \tag{12}$$

yields

$$f_L=Cr^{2L+N-2}e^{-\beta r}F(-n,2L+N-2,2\beta r). \tag{13}$$

Noting Eqs. (5) and (13), we finally obtain the following radial wave functions

$$R_{n,L}=Cr^L\exp[-\beta r]F(-n,2L+N-2,2\beta r). \tag{14}$$

In terms of Eqs. (3), (4) and (14), the bound state energy spectrum is shown below

$$E_{n,L}=-\frac{mZ^2e^4}{2\hbar^2}\frac{1}{(L+n+(N-1)/2)^2}. \tag{15}$$

Introducing the 'principal quantum' number $D=L+n+1, D=1,2,3,\ldots$ yields

$$E_{D,L}=-\frac{mZ^2e^4}{2\hbar^2}\frac{1}{(D+(N-3)/2)^2}. \tag{16}$$

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